Two methods to analyse radial diffusion ensembles: the peril of space- and time-dependent diffusion

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Summary

Radial diffusion - across drift orbits - occurs on slower timescales than diffusion of first and second adiabatic invariants, suggesting a test bed for comparing radiation belt simulations.

We build and test two analysis tools for comparing simulations - time to monotonicity t_m and mass-/energy-like moments $\mathcal{N}, \mathcal{E}$ - to (a) support future ensemble modelling, and (b) to investigate the effect of L^* dependent D_{LL}. t_m is simple and interpretable, whilst $\mathcal{N}, \mathcal{E}$ enable detailed analysis of the movement of particles and of evolution towards a low energy system.

We find several impacts of the L^* dependence, and of outer boundary choices. Initial results of $\mathcal{E}$ analysis suggest that radial diffusion may be more strongly affected by phase space density gradients than diffusion coefficients; this unexpected result needs future confirmation.

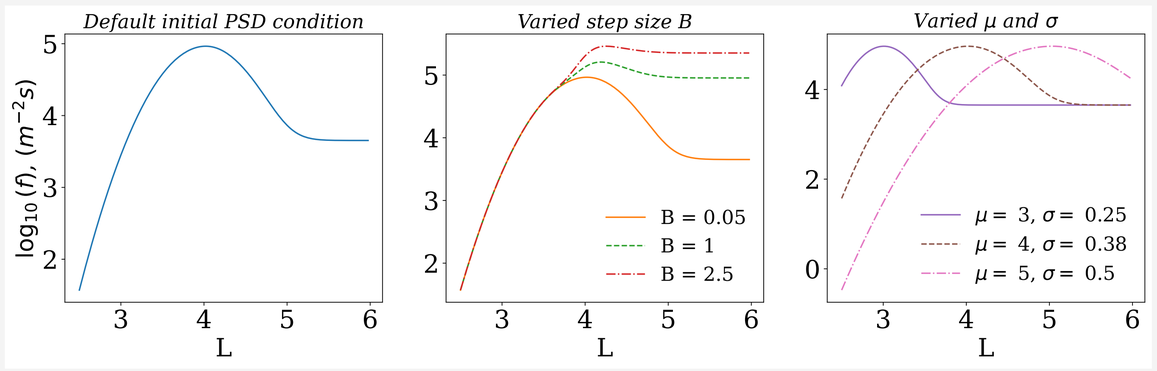
Model & Methods

We solve the 1-D radial diffusion equation for the phase space density (PSD) $f(L,t)$,

$$\frac{\partial f}{\partial t} = L^2 \frac{\partial}{\partial L}\left(\frac{D_{LL}}{L^2}\frac{\partial f}{\partial L}\right) - \mathcal{L}f,$$

with a modified Crank-Nicolson scheme ($\Delta t = 1\,\mathrm{s}$, $\Delta L = 0.1$) [2].

- **Loss** $\mathcal{L} = 1/\tau$ inside the plasmapause ($L \le L_p$) from pitch-angle scattering, using electron lifetime $\tau \approx 1.2\,E^2 L^{-1}$ [4]; zero outside.
- **Initial condition:** a Gaussian enhancement on an error-function background step, $f_0 = A\,e^{-(L-\mu)^2/2\sigma^2} + \frac{1}{2}AB\left[\mathrm{erf}(-c(L-\mu))+1\right]$.
- **Boundaries:** inner boundary fixed (Dirichlet) at $L = 2.5$; outer boundary either Neumann (zero flux) or Dirichlet (fixed value) - both physically motivated. (but we find both impose unphysical processes)
- **Two** D_{LL}**:** the fitted Ozeke $D_{LL}(L, K_p) = D_{LL}^E + D_{LL}^B$ [3], and an idealised $D_{LL} = D_0 L^n$ $(n = 6, 8)$ that makes the analytic moments tractable.



Initial PSD: default profile; varying step B; varying peak location μ and width σ.

Metrics were chosen to satisfy needs of robustness (scale independence, insensitive to threshold choices); interpretability; insight into evolution of the system.

Time to monotonicity, t_m

t_m is the time for the PSD to lose its peak and become *monotonic*, i.e. when the enhancement is no longer evident. Fig 2 shows typical PSD evolution towards monotonicity.

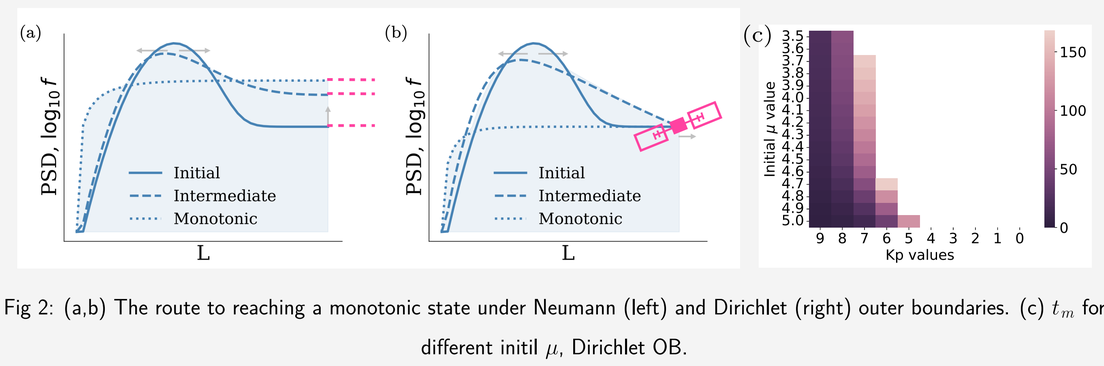


Fig 2: (a,b) The route to reaching a monotonic state under Neumann (left) and Dirichlet (right) outer boundaries. (c) t_m for different initil μ, Dirichlet OB.

- **When to use it:** as a scale-independent, interpretable timescale — "how long until an enhancement diffuses into the background." Unlike MSE-type error metrics it needs no arbitrary threshold for "finished" and is not dominated by the inner boundary.
- **Effect of plasmapause location:** when the plasmapause lies beyond the enhancement ($L_p > \mu$), pitch-angle loss carves a new local minimum, so the distribution may never become monotonic (Fig 3). A more distant L_p removes so many particles that it can change the dynamics entirely. t_m is not suitable in this case.

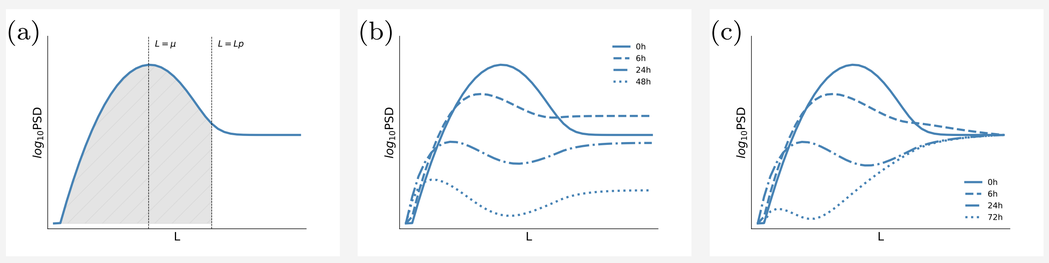


Fig 3: Monotonicity may never be reached depending on location of loss.

Analytic tools: $\mathcal{N}, \mathcal{E}$

We monitor a mass-like and an energy-like moment, each carrying the coordinate Jacobian $1/L^2$: (loss terms excluded here for simplicity)

$$\mathcal{N} = \int \frac{f}{L^2}\,dL, \qquad \mathcal{E} = \int \frac{f^2}{L^2}\,dL.$$

$\mathcal{E}$ is a positive-definite L^2-norm, minimised only when the distribution stops evolving. The rates of change separate into boundary and interior terms:

$$\mathcal{N}_t = \left[\frac{D_{LL}}{L^2} f_L\right]_{IB}^{OB}, \quad \mathcal{E}_t = \left[\frac{D_{LL}}{L^2} f f_L\right]_{IB}^{OB} - \int \frac{D_{LL}}{L^2}(f_L)^2\,dL.$$

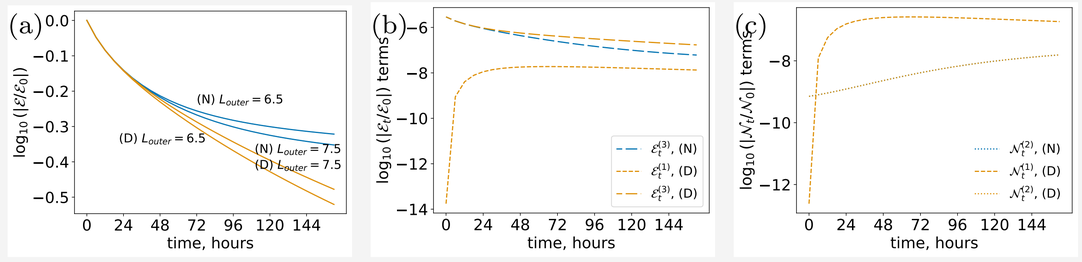
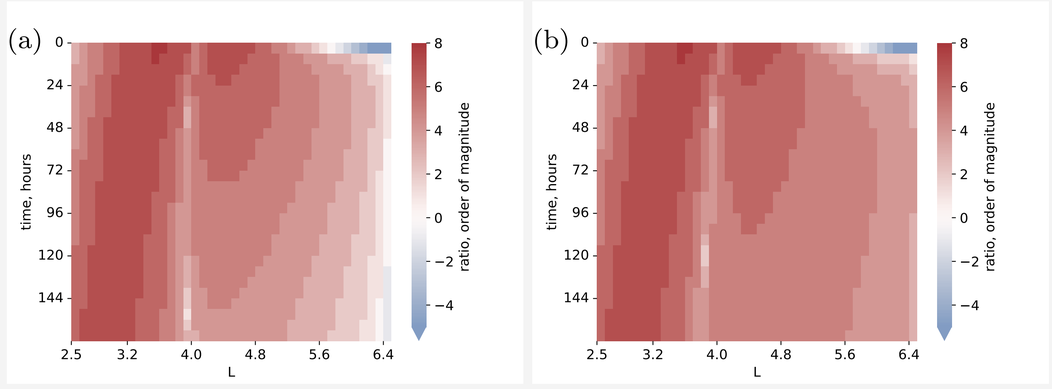


Fig 4: (a) $\mathcal{E}$ over a week, normalised by $\mathcal{E}_0$, for outer boundary set at $L^* = 6.5$ and 7.5. (b), (c) show the terms contributing to rate of change of $\mathcal{E}$ (b) and $\mathcal{N}$ (c). Neumann (blue) vs Dirichlet (orange).

- **Dependence on outer boundary location:** (Fig 4a) Evolution to a low energy state is dependent on outer boundary condition and location. The differences in evolution are reduced for a longer domain (and hence access to larger D_{LL})
- **Using** $\mathcal{E}_t, \mathcal{N}_t$**:** (Fig 4b,c)We can compare contributions of boundaries etc to loss of mass and evolution. The interior "reconfiguration" term $\mathcal{E}_t^{(3)}$ usually dominates and is the source of "diffusion acts against gradients". Non-constant D_{LL} makes this term less simple.

f_L **vs** D_{LL}**:** comparing $(f_L)^2$ against $D_0 L^{n-2}$ inside $\mathcal{E}_t^{(3)}$, the PSD *gradient* overwhelmingly dominates over D_{LL} (up to $L = 6.5$, $K_p = 4$) — contrary to the field's usual focus on improving D_{LL}. This may be significant, or a result of simplifications made so far.



Order-of-magnitude ratio $(f_L)^2 : D_0 L^{n-2}$; the gradient term dominates everywhere, for both boundary conditions.

Summary

- **Two metrics:** Time to monotonicity t_m is quick and interpretable, whilst mass- and energy-like moments $\mathcal{N}, \mathcal{E}$ enable detailed analysis of the movement of particles and the evolution towards a low energy system. t_m needs modification for some situations.
- **Metric principles:** Quantities must be integrated across the whole domain - naively averaging PSD at each L is not meaningful. Adapt other metrics with the Jacobian $1/L^2$?
- **Initial results requiring investigation:** Do gradients of the phase space density contribute more to the evolution of the system than the value of the diffusion coefficient?
- **Potential ensemble bias:** an enhancement at higher L* diffuses away more quickly; sampling μ evenly would bias an ensemble toward extra diffusion, since t_m falls non-linearly as μ rises.
- **Boundary results:** Both boundary *type* and *location* change the final PSD.
  - **Outer boundary condition:** Two different physically motivated boundary conditions result in different evolution of the system. In these idealised simulations both zero gradient and flux conditions effectively impose different unphysical processes. The difference between the two conditions reduces as simulation domain increases - but loss dominates.
  - **Domain length matters:** Curtailing the domain removes the strongest diffusion. A different L_{outer} effectively models a different physical system (although loss dominates).
  - **So what's best?** Operational models fix a Dirichlet boundary at spacecraft locations (e.g. GOES near $L \approx 6$), far short of the true belt edge ($L \approx 8$). In our experiments, **pitch-angle loss dominates over outer boundary effects**. Where data is available, short domains with data perform better than longer open-ended domains, despite the effect of domain length [5].

References

[1] Bentley et al. (2024). *Phys. Plasmas* (this work) DOI: 10.1063/5.0217268. *data, plotting tools available*
[2] Thompson et al. (2020). *JGR: Space Physics* (numerical model) DOI: 10.1029/2019JA027254.
[3] Ozeke, et al. (2014). *JGR: Space Physics* (empirical D_{LL}) DOI: 10.1002/2013JA019204.
[4] Shprits et al. (2007). *Geophys. Res. Lett.* (electron lifetime) DOI: 10.1029/2006GL029050.
[5] Lee et al. (2024). *JGR: Space Physics* (boundary experiments) 10.1029/2023JA032286

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